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Upper Secondary

# Mathematics

## Indices, Surds, Exponential and Logarithmic Equations

## Exam-Grade Questions - Word Problems

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# Change of Base Formula

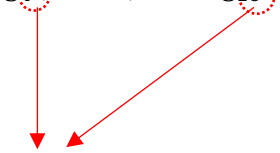
**Change of Base Formula**

More complex logarithmic equations will usually contain logarithmic terms with different bases. With the different bases, we will be unable to apply the first and second laws of logarithm (Product Law and Quotient Law). Therefore, a change of base is required.

$$log_a\, b = \frac{log_c\, b}{log_c\, a}, \qquad where\ a, b, c > 0\ and\ a, c \neq 1$$

The worked example below illustrates such a scenario.

Solve $\log_4(6-5x) - 4\log_{16} x = 1$.



Observation: The bases of the first and second term are not the same, therefore a change of base is required to change them to a common base.

$$\log_4(6-5x) - \frac{4\log_4 x}{\log_4 16} = 1$$

$$\log_4(6-5x) - \frac{4\log_4 x}{\log_4 4^2} = 1$$

$$\log_4(6-5x) - \frac{4\log_4 x}{2\log_4 4} = 1$$

$$\log_4(6-5x) - \frac{4\log_4 x}{2} = 1$$

$$\log_4(6-5x) - 2\log_4 x = 1$$

$$\log_4(6-5x) - \log_4 x^2 = 1$$

$$\log_4\left(\frac{6-5x}{x^2}\right) = 1$$

$$4^1 = \frac{6-5x}{x^2}$$

$$\vdots$$

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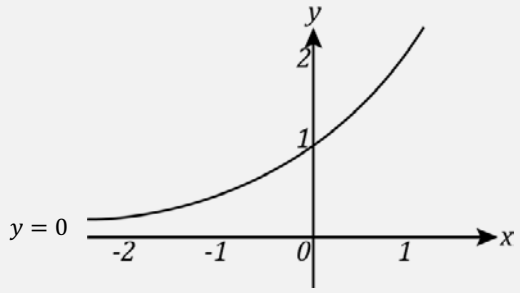
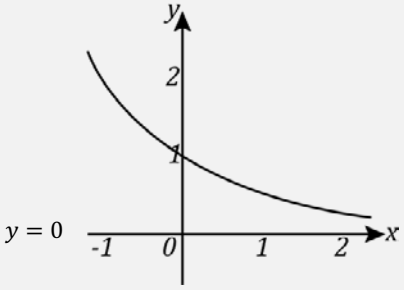
# Graphs of Exponential and Logarithm Functions

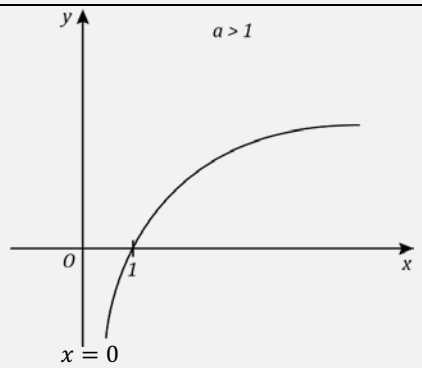
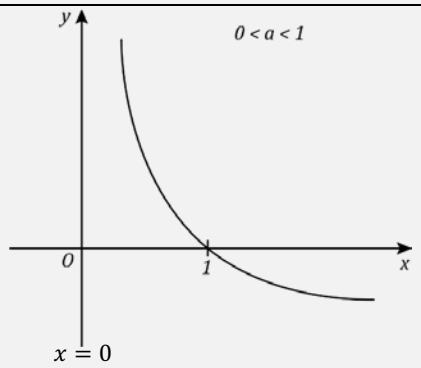
### Graphs of Exponential Functions

$$y = a^x, a \neq 1$$

| $a > 1$ | $0 < a < 1$ |
|---|---|
|  |  |
| • The graph is increasing.<br>• As $x \to -\infty, y \to 0$; as $x \to \infty, y \to \infty$. | • The graph is decreasing.<br>• As $x \to -\infty, y \to \infty$; as $x \to \infty, y \to 0$. |
| • The y-intercept is $(0, 1)$.<br>• The horizontal asymptote is $y = 0$. | |

### Graph of Logarithmic Functions

$$y = \log_a x\,, a \neq 1$$

| $a > 1$ | $0 < a < 1$ |
|---|---|
|  |  |
| • The graph is increasing.<br>• As $x \to 0, y \to -\infty$; as $x \to \infty, y \to \infty$. | • The graph is decreasing.<br>• As $x \to 0, y \to \infty$; as $x \to \infty, y \to -\infty$. |
| • The x-intercept is $(1, 0)$.<br>• The vertical asymptote is $x = 0$. | |

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# Division of Surds and Rationalising the Denominator

**Division of Surds and Rationalising the Denominator**

Consider the number $\frac{10}{\sqrt{5}}$. The denominator of the fraction is a surd. Removing the surd from the denominator is called **rationalising the denominator**. In the section, we will learn how to rationalise the denominator.

For fractions of the form $\frac{b}{p\sqrt{a}}$ or $\frac{q\sqrt{b}}{p\sqrt{a}}$, where $a$, $b$ are whole numbers and $p$, $q$ are rational, we can rationalise the denominator by multiplying the numerator and denominator by $\sqrt{a}$.

**Conjugates and how they work**

It is easy to rationalise the denominator in the examples above because the denominator consists of only a single surd term. However, what if the denominator is in the form $p + q\sqrt{b}$, such as $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{7}}$ ?

To help us rationalise fractions with denominators in the form $p + q\sqrt{b}$, such as $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{7}}$, we will need to multiply the denominator with its **conjugate surd**.

**What is a conjugate?**

Notice that $(\boldsymbol{a + b\sqrt{k}})(\boldsymbol{a - b\sqrt{k}}) = \boldsymbol{a^2 - (b\sqrt{k})^2}$

$$= \boldsymbol{a^2 - b^2 k}, \textbf{ which is rational}$$

In general, the product of $a + b\sqrt{k}$ and $a - b\sqrt{k}$ yields a rational number; we call these surds conjugate pairs. Conjugate pairs may be used to rationalise the denominator, that is, to change a fraction whose denominator contains a surd to a rational number.

Examples of conjugate pairs are $5 + 2\sqrt{3}$ and $5 - 2\sqrt{3}$, $6 - 3\sqrt{2}$ and $6 + 3\sqrt{2}$.

- To rationalise the root of $\frac{a}{\sqrt{b}}$, multiply by $\frac{\sqrt{b}}{\sqrt{b}}$.
- To rationalise the root of $\frac{a}{b+\sqrt{c}}$, multiply by $\frac{b-\sqrt{c}}{b-\sqrt{c}}$.

Choosing of the conjugate

| Denominator | Conjugate Surd |
|---|---|
| $1 + \sqrt{3}$ | $1 - \sqrt{3}$ |
| $\sqrt{7} - \sqrt{2}$ | $\sqrt{7} + \sqrt{2}$ |

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Upon taking a closer look, you will notice that the relationship between the denominator and the conjugate surd is representative of the algebraic identity $a^2 - b^2 = (a + b)(a - b)$.

For example,

| Denominator: $5 + \sqrt{3}$ | Conjugate surd: $5-\sqrt{3}$ |
|---|---|
| Denominator × Conjugate<br>$= (5 + \sqrt{3})(5 - \sqrt{3})$<br>$= (5)^2 - (\sqrt{3})^2$<br>$= 22$ | |

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# Exam Grade Questions

**Practice 1**

Given that the diagonal of a square is $\frac{1-5\sqrt{2}}{4-4\sqrt{2}}$, calculate $P^2$, where $P$ is the perimeter of this square. Give your answer in the form $a + b\sqrt{2}$ where $a$ and $b$ are real numbers.

**Practice 2**

Given a triangle $ABC$ has area $\left(\frac{1}{2} + \frac{3}{2}\sqrt{2}\right)\ m^2$ and side *AB* is $\left(2 + \sqrt{8}\right)\ m$.

Express the perpendicular distance from $C$ to $AB$ in the form $\left(a + b\sqrt{2}\right)$ where $a$ and $b$ are real numbers.

**Practice 3**

A right cylinder has a diameter of $\left(2 + 2\sqrt{2}\right)\ cm$ and a volume $\left(7 + 4\sqrt{2}\right)\pi\ cm^3$. Find, without using a calculator, the height of the cylinder in the form $\left(a + b\sqrt{2}\right)\ cm$, where $a$ and $b$ are integers.

**Practice 4**

The base radius of a solid cone is $\left(3 + \sqrt{2}\right)$ cm. Given that the volume of the cone is $\left(20 + 2\sqrt{2}\right)\pi$ cm$^3$, express the perpendicular height of the cone in the form $\left(a + b\sqrt{2}\right)$ cm, where *a* and *b* are constants.

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**Practice 5**

An equilateral triangle has sided of length $\frac{6}{2+\sqrt{3}}$.

(a) Show that the height of the triangle is $6\sqrt{3}-9$.

(b) Hence, or otherwise, find the area of the triangle in the form $p+q\sqrt{3}$, where *p* and *q* are integers.

**Practice 6**

Show that $2(9^{n+1})+3^{2n+3}-9^{n}$ is a multiple of 11 for all positive integer values of *n*.

**Practice 7**

If $a=2^{x}$ and $b=3^{x}$, express in the simplest form in terms of *a* and *b*,

(a) $\left(3\frac{3}{8}\right)^{x}$,

(b) $96^{x}-54^{x}$.

**Practice 8**

If $\frac{r^{2}}{8}(3x)^{3r}\left(\frac{2}{27x^{2}}\right)^{6-r}$ can be simplified to $\frac{p}{x^{2}}$, find the values of the constants *r* and *p*.

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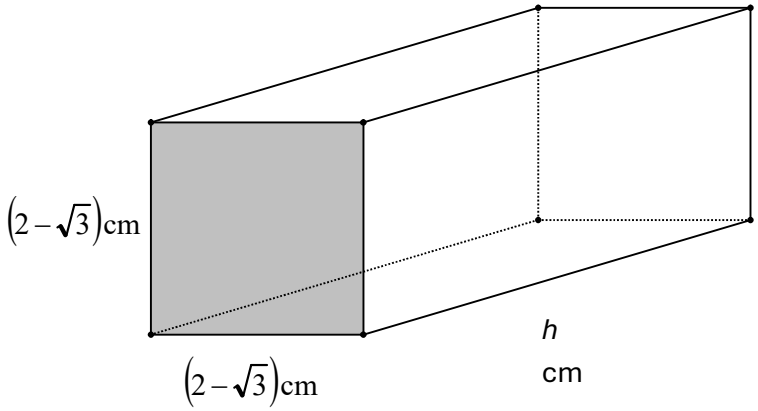
## Tutorial

1. Given that $\frac{\sqrt{x^{\frac{4}{3}}y^{-\frac{2}{5}}}}{x^{-\frac{1}{3}}y^{\frac{3}{5}}} = x^a y^b$, find the value of $a$ and of $b$.

2. Given that $u = 3^x$, express $9^{x-2}$ in terms of $u$.

3. Given that the base and perpendicular height of a parallelogram are $\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ cm and $\sqrt{6}$ cm respectively, calculate the area of this parallelogram, giving your answer in the form of $a\sqrt{b} + c$ where $a$, $b$ and $c$ are integers.

4.




The diagram shows a square prism with dimensions $(2-\sqrt{3})$ cm by $(2-\sqrt{3})$ cm by $h$ cm.

Find, without using a calculator,

(a) the exact value of the perimeter of the square.

(b) the exact value of the area of the square.

Given that the volume of the prism is $\sqrt{18} - \sqrt{6}$ cm$^3$, find, without using a calculator,

(c) the exact value of $h$, leaving your answer in the form $a\sqrt{b} + c\sqrt{d}$.

5. Given that $m = \frac{1}{2+\sqrt{3}}$ and $n = 2 + \sqrt{3}$,

(a) express $n^2$ in its simplest surd form $a + b\sqrt{3}$.

(b) rationalise $m$ into its simplest surd form $c + d\sqrt{3}$.

(c) Hence, or otherwise, find the quadratic equation in $x$ whose roots are $m$ and $n$.

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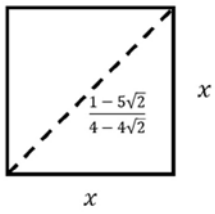
# Exam Grade Questions

**Practice 1**

**Solution**




Let the side of the square be $x$ cm.

By Pythagoras' Theorem, diagonal length$^2 = x^2 + x^2 = 2x^2$

Diagonal length$^2 = \left(\frac{1-5\sqrt{2}}{4-4\sqrt{2}}\right)^2$

$= \frac{\left(1-5\sqrt{2}\right)^2}{4^2\left(1-\sqrt{2}\right)^2}$

$= \frac{1-10\sqrt{2}+50}{16\left(1-2\sqrt{2}+2\right)}$

$= \frac{51-10\sqrt{2}}{16\left(3-2\sqrt{2}\right)} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$

$= \frac{\left(113+72\sqrt{2}\right)}{16(9-8)}$

$2x^2 = \frac{113+72\sqrt{2}}{16}$

$P^2 = (4x)^2 = 16x^2 = 8(2x^2)$

$P^2 = 8\left(\frac{113+72\sqrt{2}}{16}\right)$

$P^2 = \frac{113}{2} + 36\sqrt{2}$

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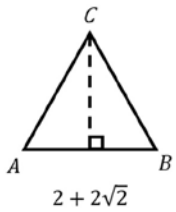
**Practice 2**

**Solution**




Let $h$ be the perpendicular distance from $C$ to $AB$.

Area of triangle $= \frac{1}{2} + \frac{3}{2}\sqrt{2}$

$$\frac{1}{2}\left(2+\sqrt{8}\right)h = \frac{1}{2}\left(1+3\sqrt{2}\right)$$

$$\left(2+2\sqrt{2}\right)h = 1+3\sqrt{2}$$

$$h = \frac{1+3\sqrt{2}}{2+2\sqrt{2}}$$

$$h = \frac{1+3\sqrt{2}}{2\left(1+\sqrt{2}\right)} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$h = \frac{-5+2\sqrt{2}}{2(1-2)}$$

$$h = \frac{5-2\sqrt{2}}{2} = \frac{5}{2} - \sqrt{2}$$

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**Practice 3**

**Solution**

$r = 1 + \sqrt{2}, V = (7 + 4\sqrt{2})\pi$

Volume of cylinder, $V = \pi r^2 h$

$(7 + 4\sqrt{2})\pi = \pi(1 + \sqrt{2})^2 h$

$h = \dfrac{7 + 4\sqrt{2}}{(1 + \sqrt{2})^2} = \dfrac{7 + 4\sqrt{2}}{1 + 2\sqrt{2} + 2}$

$h = \dfrac{7 + 4\sqrt{2}}{3 + 2\sqrt{2}} \times \dfrac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}}$

$h = \dfrac{5 - 2\sqrt{2}}{9 - 8} = 5 - 2\sqrt{2}$

**Practice 4**

**Solution**

$(20 + 2\sqrt{2})\pi = \dfrac{1}{3}\pi(3 + \sqrt{2})^2 h$

$3(20 + 2\sqrt{2}) = (9 + 6\sqrt{2} + 2)h$

$h = \dfrac{3(20 + 2\sqrt{2})}{11 + 6\sqrt{2}} \times \dfrac{11 - 6\sqrt{2}}{11 - 6\sqrt{2}}$

$h = \dfrac{3(220 - 120\sqrt{2} + 22\sqrt{2} - 24)}{121 - 72}$

$h = \dfrac{3(196 - 98\sqrt{2})}{49}$

$h = 12 - 6\sqrt{2}$

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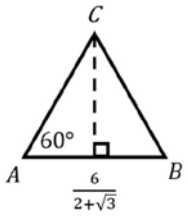
**Practice 5**

**Solution**

(a)




| Method 1: Using Special Angle | Method 2: Using Pythagoras' Theorem |
|---|---|
| $\sin 60° = \frac{h}{AC}$<br>$h = \frac{6}{2+\sqrt{3}} \times \sin 60° = \frac{6}{2+\sqrt{3}} \times \frac{\sqrt{3}}{2}$<br>$h = \frac{3\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$<br>$h = \frac{6\sqrt{3}-3(3)}{4-3}$<br>$h = 6\sqrt{3} - 9$ (Shown) | $h = \sqrt{\left(\frac{6}{2+\sqrt{3}}\right)^2 - \left(\frac{3}{2+\sqrt{3}}\right)^2}$<br>$h = \sqrt{\frac{27}{\left(2+\sqrt{3}\right)^2}}$<br>$h = \frac{\sqrt{27}}{2+\sqrt{3}}$<br>$h = \frac{3\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$<br>$h = \frac{6\sqrt{3}-3(3)}{4-3}$<br>$h = 6\sqrt{3} - 9$ (Shown) |

(b)

$$\text{Area of triangle} = \frac{\left(12-6\sqrt{3}\right)\left(6\sqrt{3}-9\right)}{2}$$
$$= 63\sqrt{3} - 108$$

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**Practice 6**

**Solution**

Method 1: Power more than 1 term -> Split by law of indices first

$2(9^{n+1}) + 3^{2n+3} - 9^n = 2(9^n)(9) + (3^2)^n(3^3) - 9^n$

$= 18(9^n) + 27(9^n) - 1(9^n)$

$= 44(9^n)$

$= 11(4)(9^n)$

Since $4(3^{2n})$ is an integer for positive integer values of $n$, the expression is a multiple of 11.

Method 2: Convert to same base, pull out common factor (smallest power)

$2(9^{n+1}) + 3^{2n+3} - 9^n$

$= 2(3^{2n+2}) + 3^{2n+3} - 3^{2n}$

$= 3^{2n}[2(3^2) + 3^3 - 1]$

$= 44(3^{2n})$

$= 11(4)(3^{2n})$

Since $4(3^{2n})$ is an integer for positive integer values of $n$, the expression is a multiple of 11.

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**Practice 7**

**Solution**

(a)

$$\left(\frac{27}{8}\right)^x = \left(\frac{3^3}{2^3}\right)^x$$
$$= \left(\frac{3^x}{2^x}\right)^3$$
$$= \left(\frac{b}{a}\right)^3$$

(b)

$$96^x - 54^x = 2^{5x}3^x - 2^x3^{3x}$$
$$= (2^x)^5 3^x - 2^x(3^x)^3$$
$$= a^5 b - ab^3$$
$$= ab(a^4 - b^2)$$
$$= ab(a^2 + b)(a^2 - b)$$

**Practice 8**

**Solution**

$$\frac{r^2}{8}(3x)^{3r}\left(\frac{2}{27x^2}\right)^{6-r} = \frac{r^2}{2^3}(3^{3r})(x^{3r})\left(\frac{2}{3^3x^2}\right)^{6-r}$$

$$= r^2(2^{3-r})\left(3^{3r-3(6-r)}\right)x^{3r-(12-2r)}$$

$$= r^2(2^{3-r})(3^{6r-18})x^{5r-12} = px^{-2}$$

$$\therefore 5r - 12 = -2$$

$$r = 2$$

$$p = 2^2(2^1)(3^{-6}) = \frac{8}{729}$$

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# Tutorial

1.

**Solution**

$$\frac{\sqrt{x^{\frac{4}{3}}y^{-\frac{2}{5}}}}{x^{-\frac{1}{3}}y^{\frac{3}{5}}}$$

$$= \frac{x^{\frac{2}{3}}y^{-\frac{1}{5}}}{x^{-\frac{1}{3}}y^{\frac{3}{5}}}$$

$$= x^{\frac{2}{3}-\left(-\frac{1}{3}\right)}y^{-\frac{1}{5}-\frac{3}{5}}$$

$$= xy^{-\frac{4}{5}}$$

$$\therefore a = 1, b = -\frac{4}{5}$$

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2.

**Solution**

$$9^{x-2} = 3^{2x-4}$$
$$= \frac{(3^x)^2}{3^4}$$
$$= \frac{u^2}{81}$$

Prove that $2^x + \frac{1}{2}(2^{x+4}) - 2^{x+2}$, where $x$ is a positive integer, is exactly divisible by 5.

$$2^x + \frac{1}{2}(2^{x+4}) - 2^{x+2} = 2^x + 2^{-1}(2^x \times 2^4) - (2^x \times 2^2)$$
$$= 2^x + 2^{-1}(2^x)(2^4) - (2^x)(2^2)$$
$$= 2^x(1) + (2^x)(2^{4-1}) - (2^x)(2^2)$$
$$= (2^x)(1 + 2^3 - 2^2)$$
$$= (2^x)(1 + 8 - 4)$$
$$= (2^x)(5)$$

$(2^x)(5)$ is multiple of 5, divisible by 5. (Proven)

<u>Alternate Method:</u>

$$2^x + \frac{1}{2}(2^{x+4}) - 2^{x+2} = 2^x + 2^{-1}(2^x \times 2^4) - (2^x \times 2^2)$$
$$= 2^x + 2^{-1}(2^x)(2^4) - (2^x)(2^2)$$

Let $y = 2^x$,

$$y(1) + (y)(2^{4-1}) - (y)(2^2) = (y)(1 + 2^3 - 2^2)$$
$$= (y)(1 + 8 - 4)$$
$$= (y)(5)$$
$$= 5y$$

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3.

**Solution**

Area of parallelogram

$$= \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \sqrt{6}$$
$$= \frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}}$$
$$= \frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$
$$= \frac{\sqrt{36}+\sqrt{24}}{4}$$
$$= 2\sqrt{6}+6 \text{ cm}^2$$

4.

**Solution**

(a)

$$\text{Perimeter} = 4 \times \left(2-\sqrt{3}\right)$$
$$= 4\left(2-\sqrt{3}\right)\text{cm}$$

(b)

$$\text{Area} = \left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)$$
$$= 4+3-4\sqrt{3}$$
$$= 7-4\sqrt{3} \text{ cm}^2$$

(c)

$$h = \frac{\sqrt{18}-\sqrt{6}}{7-4\sqrt{3}}$$
$$= \frac{\sqrt{18}-\sqrt{6}}{7-4\sqrt{3}} \times \frac{7+4\sqrt{3}}{7+4\sqrt{3}}$$
$$= \frac{\left(\sqrt{18}-\sqrt{6}\right)\left(7+4\sqrt{3}\right)}{49-48}$$
$$= \left(3\sqrt{2}-\sqrt{6}\right)\left(7+4\sqrt{3}\right)$$
$$= 21\sqrt{2}-7\sqrt{6}+12\sqrt{6}-4\sqrt{18}$$
$$= 21\sqrt{2}+5\sqrt{6}-12\sqrt{2}$$
$$= 9\sqrt{2}+5\sqrt{6} \text{ cm}$$

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5.

**Solution**

(a)

$$\begin{aligned} n^2 &= \left(2+\sqrt{3}\right)^2 \\ &= 4+4\sqrt{3}+3 \\ &= 7+4\sqrt{3} \end{aligned}$$

(b)

$$\begin{aligned} m &= \frac{1}{2+\sqrt{3}} \\ &= \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} \\ &= \frac{2-\sqrt{3}}{4-3} \\ &= 2-\sqrt{3} \end{aligned}$$

(c)

$$\begin{aligned} x^2-(m+n)x+(mn) &= 0 \\ x^2-\left(2+\sqrt{3}+2-\sqrt{3}\right)x+\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) &= 0 \\ x^2-4x+(4-3) &= 0 \\ x^2-4x+1 &= 0 \end{aligned}$$

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