INDIGO

Junior College

H2 Mathematics

Structured Revision on Vectors

(Concept Summary and Exam Grade Questions)

- Basic Formulae (Area of Δ/Projection, etc)
- Dot/Cross Product and its Properties
- Equation of Lines and their behaviours
- Equation of Planes and their behaviours

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

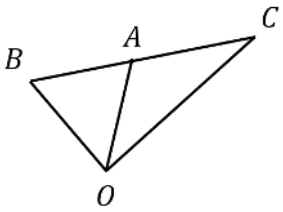
INDIGO

Copyright © 2024/25
Sale or duplication is prohibited.

Concept Summary

Basic vectors

Addition/Subtraction of Vectors



Vector Addition:

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

Vector Subtraction:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

Magnitude

Given $\overrightarrow{AP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

$$\left|\overrightarrow{AP}\right| = \sqrt{x^2 + y^2 + z^2}$$

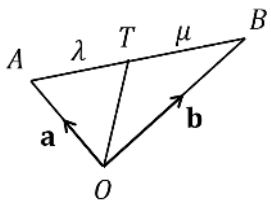
Unit vector

$$\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{12^2 + 3^2 + 4^2}} \begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix}$$

$$\hat{\mathbf{a}} = \frac{1}{13} \begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix}$$

Ratio Theorem



$$\overrightarrow{OT} = \frac{\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}}{\lambda + \mu}$$

If ratio theorem is needed but the ratio is not given, let $AT : TB = \lambda : (1 - \lambda)$, where $0 < \lambda < 1$.

Midpoint Theorem

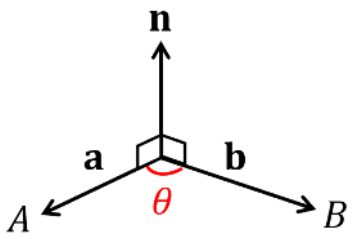
(Special case of ratio theorem, $\lambda = \mu$)

$$\overrightarrow{OT} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OA}\right)$$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

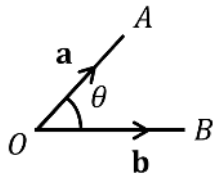
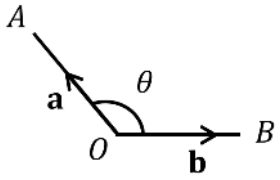
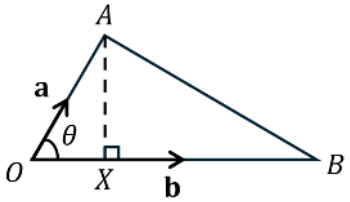
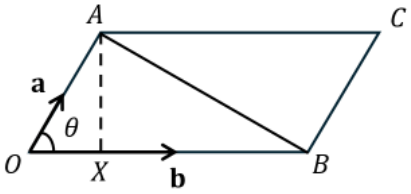
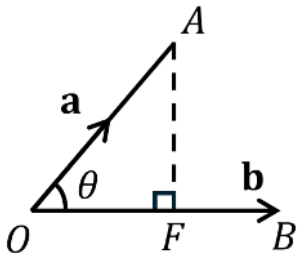
Scalar (Dot) Product & Vector (Cross) Product



	Dot Product	Cross Product
Formula	$\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}1\\1\\5\end{pmatrix}\cdot\begin{pmatrix}7\\-1\\-2\end{pmatrix}$ $=a_1b_1+a_2b_2+a_3b_3$ $=(1)(7)+(1)(-1)+(5)(-2)$ $=-4$	$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}1\\1\\5\end{pmatrix}\times\begin{pmatrix}7\\-1\\-2\end{pmatrix}$ $=\begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$ $=\begin{pmatrix}(1)(-2)-(5)(-1)\\-[(1)(-2)-(5)(7)]\\(1)(-1)-(1)(7)\end{pmatrix}$ $=\begin{pmatrix}3\\37\\-8\end{pmatrix}$
	$\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$	$\mathbf{a}\times\mathbf{b}=(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta)\hat{\mathbf{n}}$ $\lvert\mathbf{a}\times\mathbf{b}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$
	$\mathbf{a}\cdot\mathbf{b}$ gives us a **scalar (constant)**.	$\mathbf{a}\times\mathbf{b}$ gives us a **vector**.
Properties	1) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ 2) $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$ 3) $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c})=\mathbf{a}\cdot\mathbf{b}\pm\mathbf{a}\cdot\mathbf{c}$ If $\mathbf{a}\cdot\mathbf{b}=0$, $\mathbf{a}$ is perpendicular to $\mathbf{b}$	1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ 2) $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ 3) $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=\mathbf{a}\times\mathbf{b}\pm\mathbf{a}\times\mathbf{c}$ If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, $\mathbf{a}$ is parallel to $\mathbf{b}$.

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

<table>
<tr><th colspan="2">Application</th></tr>
<tr><th colspan="2">Angle between two vectors</th></tr>
<tr><td>To find $\angle AOB$,
$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$</td><td> 
</td></tr>
<tr><td>If $\mathbf{a}\cdot\mathbf{b} < 0$, θ is obtuse.
To find acute angle,
$$\cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$</td><td> 
</td></tr>
<tr><th>Area of triangle</th><th>Area of Parallelogram</th></tr>
<tr><td> 
</td><td> 
</td></tr>
<tr><td>Area of $\Delta OAB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$</td><td>Area of $OACB = |\mathbf{a}\times\mathbf{b}|$</td></tr>
<tr><th colspan="2">Projection</th></tr>
<tr><td>Length of projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$
$= |\overrightarrow{OF}|$
$= \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a}\cdot\hat{\mathbf{b}}|$</td><td> 
</td></tr>
<tr><td>Projection of vector $\overrightarrow{OA}$ onto $\overrightarrow{OB}$
$= \overrightarrow{OF}$
$= (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$

[No modulus, because $\overrightarrow{OA}$ and $\overrightarrow{OB}$ may be pointing in different directions, i.e. θ is obtuse.]</td><td>Perpendicular/Shortest distance
from A to OB
$= AF$
$= |\mathbf{a}\times\hat{\mathbf{b}}|$</td></tr>
</table>

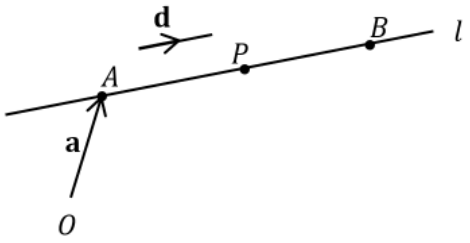
Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Lines

Equation of a Line

Given $A(2,3,1)$ and $\overrightarrow{AB} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$,



Vector (Parametric):

$l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}$

$= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

Requires:
1) Given two points A and B, or
2) Given one point A and one direction vector $\overrightarrow{AB}$.

Cartesian:
(in terms of x, y and z)

For $l : \mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} h \\ k \\ 0 \end{pmatrix}$

Make λ the subject, the cartesian equation is

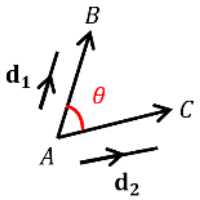
$$\frac{x-a}{h} = \frac{y-b}{k}, \qquad z = c$$

Angle between lines

$$\cos\theta = \frac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}$$

For acute angle,

$$\cos\theta = \frac{|\mathbf{d_1} \cdot \mathbf{d_2}|}{|\mathbf{d_1}||\mathbf{d_2}|}$$



Intersection of Lines

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ -6 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Step 1: Combine into single column vector and equate $l_1 = l_2$.

$$\begin{pmatrix} 2+4\lambda \\ 1-2\lambda \\ 3-\lambda \end{pmatrix} = \begin{pmatrix} 4+2\mu \\ -6+\mu \\ 4-\mu \end{pmatrix}$$

Step 2: Rearrange, and use GC to solve λ and μ.

$$4\lambda - 2\mu = -2$$
$$-2\lambda - \mu = -7$$
$$-\lambda + \mu = 1$$

From GC, $\lambda = 2, \mu = 3$

Step 3: Sub either λ or μ into l_1 or l_2 to get the intersection point.

Sub $\lambda = 2$ into l_1 or Sub $\mu = 3$ into l_2,

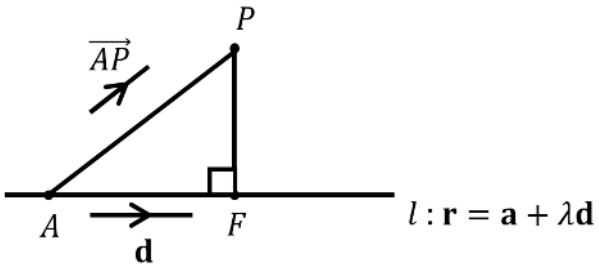
$$\mathbf{r} = \begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix}$$

$\therefore$ The point of intersection is $(10, -3, 1)$.

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

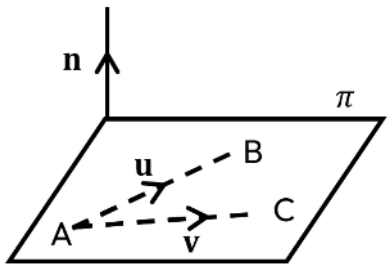
Behaviour of Lines	
For two lines, $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1}, \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \mathbf{b} + \mu\mathbf{d_2}, \mu \in \mathbb{R}$	
Step 1: To check parallel,	
Parallel: $\mathbf{d_1} = k\mathbf{d_2}$ for some $k \in \mathbb{R}$	Non-parallel: $\mathbf{d_1} \neq k\mathbf{d_2}$ for all $k \in \mathbb{R}$
Step 2: If not parallel, check for intersection	
Equate $l_1 = l_2$ to solve for λ and μ.	
If no unique solution, we have **skew lines.**	If there is unique solution, we have **intersecting lines.** Sub λ or μ into l_1 or l_2 to find the point of intersection.

From a Point to Line	
	
To find the foot of perpendicular, F	
Method 1: Algebraic **Step 1:** F is on the line l Let $\overrightarrow{OF} = a + \lambda d$ for some $\lambda \in \mathbb{R}$ **Step 2:** Find $\overrightarrow{PF}$ in terms of λ, $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP}$ **Step 3:** Since $\overrightarrow{PF}$ is $\perp$ to l, $\overrightarrow{PF} \cdot d = 0$ $\Rightarrow$ Solve for λ **Step 4:** Substitute λ into Step 1 to find $\overrightarrow{OF}$	**Method 2: Projection** **Step 1:** Project $\overrightarrow{AP}$ onto line l for vector $\overrightarrow{AF}$ **Step 2:** Use vector addition, $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$
To find perpendicular/shortest distance	
Method 1: If $\overrightarrow{PF}$ or $\overrightarrow{OF}$ is already found Distance $= \lvert\overrightarrow{PF}\rvert$	**Method 2: Projection $\overrightarrow{AP}$ onto $\overrightarrow{PF}$** Distance $= \lvert\overrightarrow{AP} \times \mathbf{d}\rvert$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Planes

<table>
<tr><th colspan="3">Equation of Plane</th></tr>
<tr><td colspan="2">Given $A(2,3,1)$, $\overrightarrow{AB}=\begin{pmatrix}1\\-3\\2\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$.</td><td> 
</td></tr>
<tr><td>Vector (Parametric) form</td><td>$\pi:\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v}, \qquad \lambda,\mu\in\mathbb{R}$
$\pi:\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix}+\mu\begin{pmatrix}3\\-2\\1\end{pmatrix}$</td><td>Condition you need:
1) Given three points, or
2) One point + two parallel vectors.</td></tr>
<tr><td>Scalar form</td><td colspan="2">Step 1: Find normal vector, $\mathbf{n}=\mathbf{p}\times\mathbf{q}$
Step 2: Apply formula:
$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$
where $\mathbf{a}$ is the position vector of the given point on the plane.
Step 3: Equation of plane,
$\pi:\ \mathbf{r}\cdot\mathbf{n}=D$</td></tr>
<tr><td>Cartesian form</td><td colspan="2">$ax+by+cz=D$ for some constant D</td></tr>
<tr><th colspan="3">Conversion</th></tr>
<tr><td>Vector form</td><td colspan="2">$\pi:\mathbf{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\5\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\3\end{pmatrix}, \qquad \lambda,\mu\in\mathbb{R}$</td></tr>
<tr><td>⇓</td><td colspan="2">$\mathbf{n}=\begin{pmatrix}-1\\0\\5\end{pmatrix}\times\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}5\\13\\1\end{pmatrix}$
$D=\mathbf{a}\cdot\mathbf{n}=\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}5\\13\\1\end{pmatrix}=15$</td></tr>
<tr><td>Scalar form</td><td colspan="2">$\pi:\mathbf{r}\cdot\begin{pmatrix}5\\13\\1\end{pmatrix}=15$</td></tr>
<tr><td>⇓</td><td colspan="2">Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\qquad\qquad \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}5\\13\\1\end{pmatrix}=15$</td></tr>
<tr><td>Cartesian form</td><td colspan="2">$\pi:5x+13y+z=15$</td></tr>
</table>

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

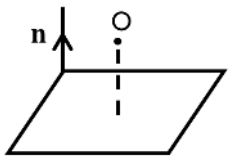
Copyright © 2024/25
Sale or duplication is prohibited.

Behaviour of Plane with Plane

Let $\pi_1 : \mathbf{r} \cdot \mathbf{n} = D$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n_2} = D_2$ where $\mathbf{n} = k\mathbf{n_2}$

Perpendicular/Shortest Distance

From origin to Plane

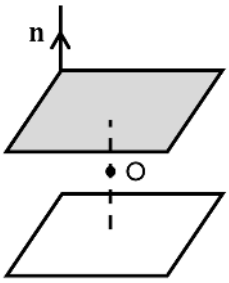


Using length of projection,

Distance from origin

$$= |\mathbf{a} \cdot \hat{\mathbf{b}}| = \frac{|\mathbf{a} \cdot \lambda\mathbf{n}|}{|\lambda\mathbf{n}|} = \frac{|\lambda||\mathbf{a} \cdot \mathbf{n}|}{|\lambda||\mathbf{n}|} = \frac{|D|}{|\mathbf{n}|}$$

Two Parallel Planes



Convert to the same normal first,

$\pi_2 : \mathbf{r} \cdot \mathbf{n_2} = D_2 \Rightarrow \mathbf{r} \cdot k\mathbf{n_2} = kD_2$

$\Rightarrow \pi_2 : r \cdot \mathbf{n} = D_3$

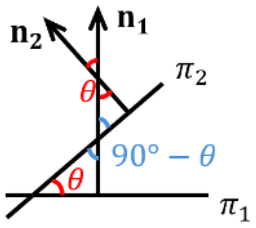
Distance between 2 parallel planes

$$= \frac{|D_3 - D|}{|\mathbf{n}|}$$

Non-parallel Planes

For $\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} = -4$ and $\pi_2 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -8$

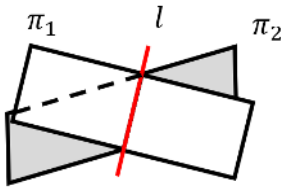
Angle Between Planes



For acute θ,

$$\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|} = |\widehat{\mathbf{n_1}} \cdot \widehat{\mathbf{n_2}}|$$

Line of Intersection Between Planes



Step 1: Convert to Cartesian

$$\pi_1 : 2x - 2y + 5z = -4$$
$$\pi_2 : x + 2y - 2z = -8$$

Step 2: Use GC to solve

$$x_1 = -4 - x_3$$
$$x_2 = -2 + \frac{3}{2}x_3$$
$$x_3 = x_3$$

Step 3: Let x_3 be k.

$$\mathbf{r} = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix} + k\begin{pmatrix} -1 \\ \frac{3}{2} \\ 1 \end{pmatrix}, k \in \mathbb{R}$$

$$\therefore l : \mathbf{r} = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}, \quad \lambda = \frac{1}{2}k \in \mathbb{R}$$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

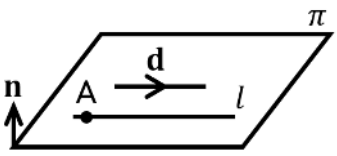
Copyright © 2024/25
Sale or duplication is prohibited.

Behaviour of Line and Plane

Let $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\overrightarrow{OA} = \mathbf{a}$ and $\pi : \mathbf{r} \cdot \mathbf{n} = D$

Line parallel to Plane

Coincident



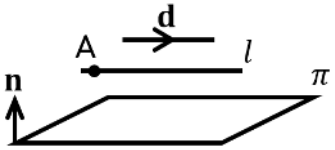
Since A is on π (a satisfies equation of π),

$$\mathbf{a} \cdot \mathbf{n} = D$$

Since l is perpendicular to n,

$$\mathbf{d} \cdot \mathbf{n} = 0$$

Non-coincident



Since A is not on π,

$$\mathbf{a} \cdot \mathbf{n} \neq D$$

Since l is perpendicular to n,

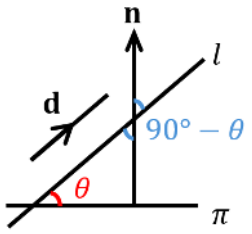
$$\mathbf{d} \cdot \mathbf{n} = 0$$

Line Not parallel to Plane

Angle between Line and Plane

For acute angle, since $\cos(90° - \theta) = \sin\theta$,

$$\sin\theta = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = |\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}|$$



Intersection between Line and Plane

Given $l : \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\pi : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 1$

Step 1: Rewrite the equation of line as single column vector

$$\mathbf{r} = \begin{pmatrix} 3 + \lambda \\ -2 - \lambda \\ 7 + 2\lambda \end{pmatrix}$$

Step 2: Substitute into equation of plane and solve λ.

$$\begin{pmatrix} 3 + \lambda \\ -2 - \lambda \\ 7 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 1$$

$$\lambda = -3$$

Step 3: Substitute $\lambda = -3$ into the line,

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

∴ The point of intersection is $(0, 1, 1)$.

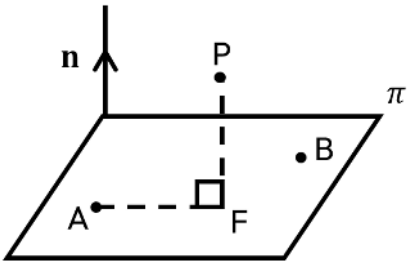
Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Point to Plane

For point $P(2,-1,0)$ and plane $\pi : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$

Foot of perpendicular



Method 1: Algebraic

Step 1: Find equation of l_{PF}

$$l_{PF} : \mathbf{r} = \overrightarrow{OP} + \lambda \mathbf{n}$$

$$l_{PF} : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

Step 2: Substitute l_{PF} into π, solve for λ.

$$\begin{pmatrix} 2-\lambda \\ -1+\lambda \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$$

$$\Rightarrow \quad \lambda = 2$$

Step 3: Substitute λ into l_{PF}

$$\overrightarrow{OF} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\therefore F(0,1,2)$$

Method 2: Projection

Step 1: Guess a point on the plane

$$\overrightarrow{OB} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Step 2: Project $\overrightarrow{PB}$ onto $\overrightarrow{PF}$

$$\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PF} = (\overrightarrow{PB} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

$$= \left[\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

Step 3: Find $\overrightarrow{OF}$

$$\overrightarrow{OF} = \overrightarrow{OP} + \overrightarrow{PF} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\therefore F(0,1,2)$$

Perpendicular distance from point to plane

Method 1: If point F is already found

$$|\overrightarrow{AF}| = |\overrightarrow{OF} - \overrightarrow{OA}|$$

Method 2: Projection $\overrightarrow{AB}$ onto $\overrightarrow{AF}$

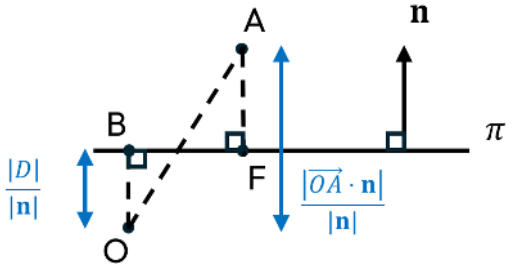
$$|\overrightarrow{AF}| = |\overrightarrow{AB} \cdot \hat{\mathbf{n}}|$$

Method 3: Formula

Distance

$$= \frac{|\overrightarrow{OA} \cdot \mathbf{n} - D|}{|\mathbf{n}|}$$

(similar to finding distance between 2 planes)



Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Exam Grade Questions

Practice 1 **[Level: Intermediate]**

The position vectors of A, B and C referred to a point O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point N is on AB such that $AN : NB = 2 : 1$. The point M is on BC such that $5BM = 3BC$.

(a) If O is the midpoint of CN, prove that $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$. [2]

(b) Show that A, O and M are collinear, and find the ratio $AO : OM$. [3]

(c) If the point P is such that $\overrightarrow{NP} = \overrightarrow{AM}$, show that the ratio of the area of $PNAM$: area of $PNOM = 12 : 7$. [3]

Solution

(a)

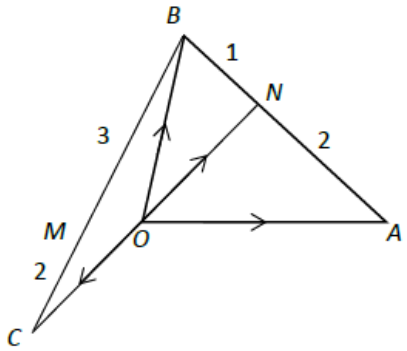
By ratio theorem, $\overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

Since O is the midpoint of CN, $\overrightarrow{ON} = -\mathbf{c}$

$$\Rightarrow \overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = -\mathbf{c}$$

$$\mathbf{a} + 2\mathbf{b} = -3\mathbf{c}$$

$\therefore \mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$ (Shown)



(b)

By ratio theorem,

$$\overrightarrow{OM} = \frac{2\mathbf{b} + 3\mathbf{c}}{2 + 3} = \frac{-\mathbf{a}}{5} = -\frac{1}{5}\overrightarrow{OA}$$

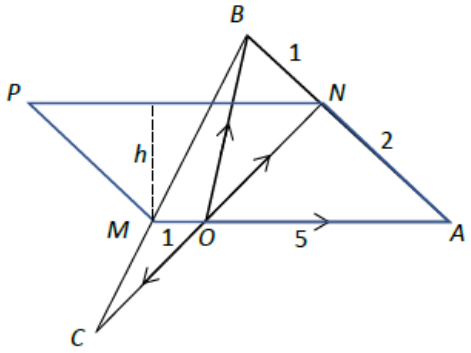
$\therefore A$, O and M are collinear.

Since $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$, the ratio $AO : OM = 5 : 1$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

(c)



Method 1:	Method 2:
Area of parallelogram $= h\lvert\overrightarrow{AM}\rvert$, where h is the height of parallelogram Area of trapezium $PNOM$ $= \frac{1}{2}(\lvert\overrightarrow{PN}\rvert + \lvert\overrightarrow{OM}\rvert)h$ $= \frac{1}{2}\left(\lvert\overrightarrow{AM}\rvert + \frac{1}{6}\lvert\overrightarrow{AM}\rvert\right)h$ $= \frac{7}{12}h\lvert\overrightarrow{AM}\rvert$ $= \frac{7}{12} \times$ Area of parallelogram $PNAM$ $\therefore$ Ratio of area of $PNAM$: area of $PNOM$ $= 12 : 7$ (Shown)	Area of parallelogram $PNAM$ $= \lvert\overrightarrow{PN} \times \overrightarrow{PM}\rvert$ $= \left\lvert\frac{6}{5}\mathbf{a} \times \left(\frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\right)\right\rvert$ $= \frac{4}{5}\lvert\mathbf{a} \times \mathbf{b}\rvert$ Area of trapezium $PNOM$ $= \lvert\overrightarrow{PN} \times \overrightarrow{PM}\rvert - \frac{1}{2}\lvert\overrightarrow{ON} \times \overrightarrow{OA}\rvert$ $= \frac{4}{5}\lvert\mathbf{a} \times \mathbf{b}\rvert - \frac{1}{2}\left\lvert\left(\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\right) \times \mathbf{a}\right\rvert$ $= \frac{4}{5}\lvert\mathbf{a} \times \mathbf{b}\rvert - \frac{1}{2}\left\lvert-\frac{2}{3}\mathbf{a} \times \mathbf{b}\right\rvert$ $= \frac{7}{15}\lvert\mathbf{a} \times \mathbf{b}\rvert$ $\therefore$ Ratio of area of $PNAM$: area of $PNOM$ $= \frac{4}{5} : \frac{7}{15} = 12 : 7$ (Shown)

Concepts	What to think about
Ratio Theorem	"Midpoint" and "Ratio of AB : BC" are the **keywords** to lookout for, when it comes to Ratio Theorem.
Collinearity	To prove collinearity, we need to show they are **parallel** and have a **common point**. e.g. A, B, C are collinear $\Rightarrow \overrightarrow{AB} = k\overrightarrow{BC}$ with common point B.
Area of parallelogram or triangle	To find ratio of area, either look out for common perpendicular height or apply cross product formula

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Practice 2 **[Level: Intermediate]**

The plane π_1 contains the point $A(1, 2, -1)$ and the line l with equation $\frac{x-1}{2} = \frac{2-z}{3}, y = -1$.

The plane π_2 contains the point $B(-5.5, 3, 2)$ and meets π_1 in the line l.

(a) Find the equation of π_1 in scalar product form. [3]

(b) Show that the vector $\overrightarrow{BF}$ is $\begin{pmatrix} 4.5 \\ -4 \\ 3 \end{pmatrix}$, where F is the foot of perpendicular from B to l. [3]

(c) Find the exact value of the shortest distance from B to π_1. [2]

(d) Hence or otherwise find the acute angle between π_1 and π_2, giving your answer to the nearest $0.1°$. [3]

Solution

(a)

$$l : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}. \lambda \in \mathbb{R}$$

Let C be a point on l such that $\overrightarrow{OC} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$$

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 3 - 2 + 4 = 5$$

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 5$$

(b)

$$\overrightarrow{OF} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \begin{pmatrix} 6.5 + 2\lambda \\ -4 \\ -3\lambda \end{pmatrix}$$

$$\overrightarrow{BF} \cdot \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = 0 \Rightarrow 13 + 4\lambda + 9\lambda = 0$$

$$\lambda = -1$$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

$\therefore \overrightarrow{BF} = \begin{pmatrix} 6.5-2 \\ -4 \\ -3(-1) \end{pmatrix} = \begin{pmatrix} 4.5 \\ -4 \\ 3 \end{pmatrix}$ (Shown)

(c)

Shortest distance from B to π_1

= Length of projection of $\overrightarrow{BF}$ onto $\mathbf{n_1}$

$= \left|\overrightarrow{BF} \cdot \widehat{\mathbf{n_1}}\right| = \dfrac{\begin{pmatrix} 4.5 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{17}} = \dfrac{23}{2\sqrt{17}}$ units

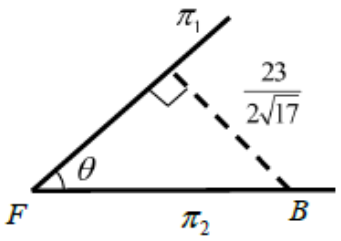
(d)

Method 1:

Let θ be acute angle between π_1 and π_2

$\sin\theta = \dfrac{\frac{23}{2\sqrt{17}}}{BF} = \dfrac{23}{2\sqrt{17}\sqrt{4.5^2+4^2+3^2}}$

$\theta = 24.5°$ (1 dp)



Method 2:

$\mathbf{n_2} = \overrightarrow{BF} \times \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 4.5 \\ -4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ 19.5 \\ 8 \end{pmatrix}$

Let θ be the acute angle between π_1 and $\frac{23}{2\sqrt{17}}$

$\cos\theta = \dfrac{\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 19.5 \\ 8 \end{pmatrix}}{\sqrt{17}\sqrt{588.25}} = \dfrac{91}{\sqrt{17}\sqrt{588.25}}$

$\theta = 24.5°$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Concepts	What to think about
Equation of plane	Given a point and a line, to find scalar form, we need normal vector. How do we find the normal vector?
Foot of perpendicular	From a point to line, we can use either projection method or algebraic method.
Shortest distance to the plane	To find perpendicular distance, we can use either projection method or algebraic method.
Angle between two planes	Formula: $\cos\theta = \frac{\lvert\mathbf{n_1}\cdot\mathbf{n_2}\rvert}{\lvert\mathbf{n_1}\rvert\lvert\mathbf{n_2}\rvert}$ Do we have both normal vectors of the two planes?

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Practice 3 **[Level: Intermediate]**

In the study of light, a straight line is used to model a ray of light. A scientist wishes to conduct an experiment using light rays.

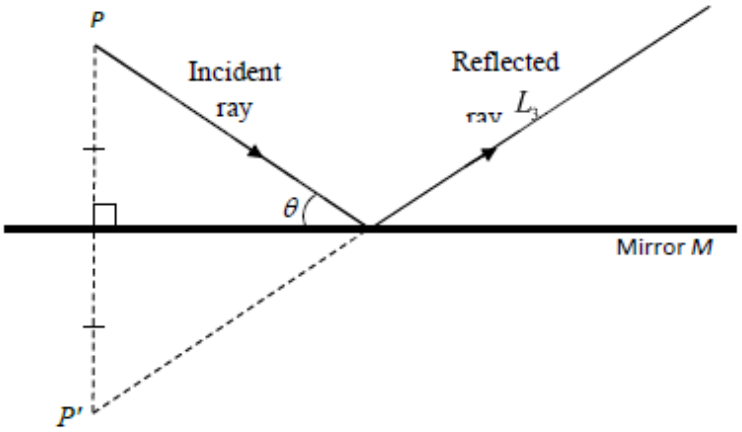
Two light rays L_1 and L_2 have equations as follows.

$$L_1 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + s(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$
$$L_2 : \mathbf{r} = 5\mathbf{j} - \frac{9}{2}\mathbf{k} + t(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}),$$

where s and t are parameters.

(a) Show that lines L_1 and L_2 are skew lines. [2]

A scientist uses a light source to shine an incident ray of light from the point P with coordinates $(1, 5, 0)$ such that it makes an angle of incidence, θ, with a plane mirror M, as shown in the diagram below. M is modelled by a plane with equation $y - z - 2 = 0$. The scientist wishes to find θ such that the reflected light ray from the mirror, denoted by L_3, cuts both the skew lines L_1 and L_2.



(b) Find the coordinates of P', the reflection of point P in M. [3]

(c) Find the cartesian equation of the plane Π containing point P' and line L_1. [2]

(d) Find the coordinates of the point of intersection of the plane Π and line L_2. [2]

(e) Hence find the cartesian equation of the reflected ray, line L_3, that cuts both the lines L_1 and L_2. [2]

(f) Find the angle of incidence, θ, that the scientist should use. [1]

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Solution

(a)

$$L_1 : \mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\2\end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix}0\\5\\-9/2\end{pmatrix} + t\begin{pmatrix}2\\-2\\3\end{pmatrix}$$

Since $\begin{pmatrix}-2\\1\\2\end{pmatrix} \neq k\begin{pmatrix}2\\-2\\3\end{pmatrix}$

L_1 is not parallel to L_2.

Equate $L_1 = L_2$

$$\begin{pmatrix}1-2s\\2+s\\2s\end{pmatrix} = \begin{pmatrix}2t\\5-2t\\9\\-\frac{2}{2}+3t\end{pmatrix}$$

$-2s - 2t = -1$ ---- (1)

$s + 2t = 3$ ---- (2)

$2s - 3t = -\frac{9}{2}$ ---- (3)

From GC, no unique solution

$\therefore L_1$ and L_2 do not intersect, they are skew lines.

(b)

Let F be the foot of perpendicular from P to M.

Equation of line through P and F:

$$\mathbf{r} = \begin{pmatrix}1\\5\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\5+\lambda\\-\lambda\end{pmatrix}$$

$$\begin{pmatrix}1\\5+\lambda\\-\lambda\end{pmatrix} \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} = 2$$

$5 + 2\lambda = 2$

$\lambda = -\frac{3}{2}$

$$\overrightarrow{OF} = \begin{pmatrix}1\\5+\lambda\\-\lambda\end{pmatrix} = \begin{pmatrix}1\\7/2\\3/2\end{pmatrix} = \frac{1}{2}\left(\overrightarrow{OP'} + \overrightarrow{OP}\right)$$

$$\overrightarrow{OP'} = 2\overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Coordinates of P' are $(1, 2, 3)$.

[**Alternative method** to find foot of perpendicular: Observe that $Q(0, 2, 0)$ is on plane M. Project $\overrightarrow{PQ}$ onto normal to find $\overrightarrow{PF} = \left|\overrightarrow{PQ} \cdot \hat{\mathbf{n}}\right|$, then $\overrightarrow{OF} = \overrightarrow{OP} + \overrightarrow{PF}$]

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

(c)

$\overrightarrow{OP'} - \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix}$ is a direction parallel to plane Π.

Normal of plane Π:

$$\mathbf{n} = \begin{pmatrix}-2\\1\\2\end{pmatrix} \times \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}1\\2\\0\end{pmatrix}$$

Equation of plane containing P' and L_1:

$$\mathbf{r} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} = 5$$

$\therefore$ Equation of $\Pi : x + 2y = 5$

(d)

$$L_2 : \mathbf{r} = \begin{pmatrix}0\\5\\-9/2\end{pmatrix} + t\begin{pmatrix}2\\-2\\3\end{pmatrix} = \begin{pmatrix}2t\\5-2t\\-\frac{9}{2}+3t\end{pmatrix}$$

Sub L_2 into Π,

$2t + 2(5 - 2t) = 5$

$t = \frac{5}{2}$

$$\Rightarrow r = \begin{pmatrix}2t\\5-2t\\-\frac{9}{2}+3t\end{pmatrix} = \begin{pmatrix}5\\0\\3\end{pmatrix}$$

Point of intersection is $(5, 0, 3)$.

(e)

Let Q be $(5, 0, 3)$

Q is on L_2, since $P'Q$ and L_1 are coplanar and are not parallel, they will intersect. Line $P'Q$ is the reflected ray L_3 that intersects the two skew lines.

Vector equation of $P'Q$, L_3 :

$$r = \begin{pmatrix}5\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}, \lambda \in \mathbb{R}$$

Cartesian equation:

$$\frac{x-5}{2} = \frac{y-0}{-1}, z = 3$$

$$\therefore \frac{x-5}{2} = -y, z = 3$$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Angle of incidence is the angle between L_3 and mirror

$$\sin\theta = \frac{\left|\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}\right|}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$$

$\theta = 18.4°$ or 0.322 radian

Concepts	What to think about
Skew Lines	Skew lines are non-intersecting and non-parallel lines.
Reflection of a point	The foot of perpendicular is the midpoint of the point and its image, and its on the reflection line of symmetry.
Intersection between line and plane	The position vector of R, $\mathbf{r}$, satisfies both equation of line and plane. How do we combine them?
Angle between line and plane	Apply formula using the direction and normal vector.

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Practice 4 **[Level: Advanced]**

Referred to the origin, the points A and B have position vectors $-\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+\mathbf{k}$ respectively. The plane π has equation $r=\begin{pmatrix}5\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\2\\a\end{pmatrix}$, and the line l has equation $r=\begin{pmatrix}3\\0\\a\end{pmatrix}+t\begin{pmatrix}4a\\4\\1\end{pmatrix}$, where a is a constant and λ, μ and t are parameters.

(a) Show that for all real values of a, l is parallel to π. [2]

(b) Find the value of a such that l and π have common points. [2]

For the rest of the question, let $a=1$.

(c) Find the projection of $\overrightarrow{AB}$ onto π. [3]

(d) Let F be the foot of perpendicular from A to π. The point C lies on AF extended such that $\angle ABF=\angle CBF$. Find a cartesian equation of the plane that contains C and l. [3]

(e) Let D be a point on l. Find the largest possible value of the non-reflex angle $\angle ADC$. [2]

Solution

(a)

Normal of π, $\mathbf{n}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}1\\2\\a\end{pmatrix}=\begin{pmatrix}2\\-(2a+1)\\4\end{pmatrix}$

$\mathbf{d}\cdot\mathbf{n}=\begin{pmatrix}4a\\4\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-2a-1\\4\end{pmatrix}=8a-8a-4+4=0$

Since $\mathbf{d}\perp\mathbf{n}$, then l is parallel to π for all real values of a.

(b)

Equation of π : $\mathbf{r}\cdot\begin{pmatrix}2\\-2a-1\\4\end{pmatrix}=\begin{pmatrix}5\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2\\-2a-1\\4\end{pmatrix}=10$

Since l is parallel to π, we want l to lie inside π.

$\begin{pmatrix}3\\0\\a\end{pmatrix}\cdot\begin{pmatrix}2\\-2a-1\\4\end{pmatrix}=10$

$6+4a=10$

$\therefore a=1$

(c)

Since B lies on the line, required vector is the vector $\overrightarrow{FB}$, where F is the foot of perpendicular from A to π.

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

$\overrightarrow{OF} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} + k\begin{pmatrix}2\\-3\\4\end{pmatrix}$ for some $k \in \mathbb{R}$

Sub $\overrightarrow{OF}$ into π,

$\begin{pmatrix}-1+2k\\3-3k\\-2+4k\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\4\end{pmatrix} = 10$

$-19 + 29k = 10$

$k = 1$

$\overrightarrow{OF} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\overrightarrow{FB} = \begin{pmatrix}3\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$

(d)

Let π_2 be the required plane.

Point C is the reflection of A in π_1.

$\overrightarrow{OC} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} + 2\begin{pmatrix}2\\-3\\4\end{pmatrix} = \begin{pmatrix}3\\-3\\6\end{pmatrix}, \quad \overrightarrow{BC} = \begin{pmatrix}3\\-3\\6\end{pmatrix} - \begin{pmatrix}3\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-3\\5\end{pmatrix}$

Normal of π_2, $\mathbf{n} = \mathbf{d} \times \overrightarrow{BC} = \begin{pmatrix}4\\4\\1\end{pmatrix} \times \begin{pmatrix}0\\-3\\5\end{pmatrix} = \begin{pmatrix}23\\-20\\-12\end{pmatrix}$

$\mathbf{r} \cdot \begin{pmatrix}23\\-20\\-12\end{pmatrix} = \begin{pmatrix}3\\0\\1\end{pmatrix} \cdot \begin{pmatrix}23\\-20\\-12\end{pmatrix} = 57$

$\therefore$ Equation of π_2 is $23x - 20y - 12z = 57$

(e)

For maximum $\angle ADC$, DF is the shortest distance from F to l

$\Rightarrow \max \angle ADC = 2 \times \angle CDF$

$= 2 \times \cos^{-1} \dfrac{\begin{pmatrix}2\\-3\\4\end{pmatrix} \cdot \begin{pmatrix}23\\-20\\-12\end{pmatrix}}{\sqrt{29}\sqrt{1073}}$

$= 2 \times \cos^{-1} \dfrac{58}{\sqrt{29}\sqrt{1073}}$

$= 141.6°$ (1dp)

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Concepts	What to think about
Line in plane	For a line in plane, we can either **show the line satisfies π for all real values of λ**, or **we show that they are parallel and have common point**.
Projection vector on a plane	Sketch if necessary. Does A or B lie on the plane? Let F be the foot of perpendicular, are we finding $\overrightarrow{FA}$ or $\overrightarrow{FB}$?
Largest angle between two lines	To find the largest angle, point D must be nearest to the foot of perpendicular of the line AC.

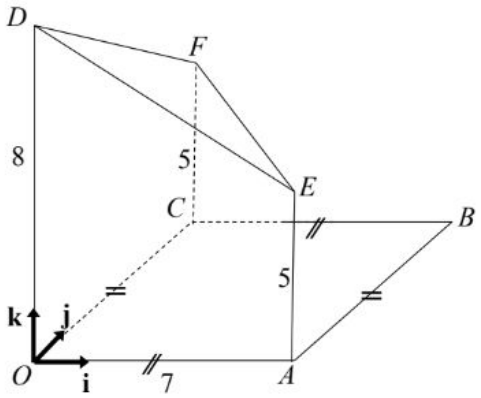
Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

Quiz

Q1. [25min]

The diagram (not drawn to scale) shows the structure of a partially constructed building that is built on a horizontal ground. The building has a square base foundation of 7 m in length. Points O, A, B and C are the corners of the foundation of the building. The building currently consists of three vertical pillars OD, AE and CF of heights 8 m, 5 m and 5 m respectively. A canvas is currently attached at D, E and F, forming a temporary shelter for the building. O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, each of length 1 m, are taken along OA, OC and OD respectively.



(a) Find a Cartesian equation of the plane that represents the canvas DEF. [3]

(b) Find the acute angle which the canvas DEF makes with the horizontal ground. [2]

(c) Given that the canvas is to be extended along the plane DEF till it touches the horizontal ground, explain why the point B will lie beneath the canvas. [2]

A cement roof is to be built to replace the extended canvas. A vertical partition wall is also to be built such that it is d m away from and parallel to the plane $ODFC$, where $0 < d < 7$.

(d) Find the exact vector equation of the line where the roof meets the partition wall. Show your working clearly, leaving your answer in terms of d. [4]

(e) A lighting point, P, is to be placed on the roof such that it is closest to B. Find the position vector of P. [3]

Solution

(a)

$$\overrightarrow{OD} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}, \quad \overrightarrow{OE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}, \quad \overrightarrow{OF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}$$

$$\overrightarrow{DE} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}, \overrightarrow{DF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}, \overrightarrow{EF} = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 0 \end{pmatrix}$$

A vector perpendicular to the plane is

$= \overrightarrow{DE} \times \overrightarrow{DF}$

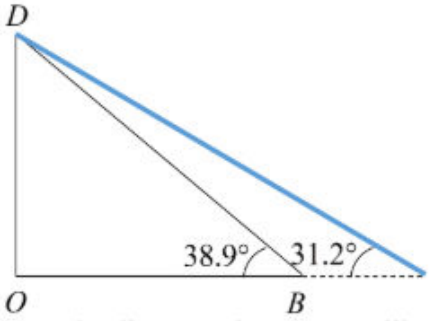
Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

$= \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 21 \\ 21 \\ 49 \end{pmatrix} = 7\begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$
Cartesian equation of the plane is
$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56$
$\therefore 3x + 3y + 7z = 56$

(b)
Let the required angle be θ.
$\cos\theta = \dfrac{\left|\begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\sqrt{3^2 + 3^2 + 7^2}\sqrt{1}} = \dfrac{7}{\sqrt{67}}$
$\theta = 31.2°$ (1 dp) or 0.545 radian

(c)

Method 1:	Method 2:
$\left\|\overrightarrow{OB}\right\| = \sqrt{7^2 + 7^2} = \sqrt{98}$ $\left\|\overrightarrow{OD}\right\| = 8$ $\tan\angle OBD = \dfrac{8}{\sqrt{98}}$ $\angle OBD = 38.9°$  From the diagram, the canvas will cover B.	$\overrightarrow{OB} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}$ Equation of perpendicular line passing through B is $l : \mathbf{r} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ Sub l into plane DEF, $\begin{pmatrix} 7 \\ 7 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56$ $\lambda = 2 > 0$, l and plane DEF will intersect above the horizontal ground. So the canvas will cover point B.
Method 3:	
Consider a plane parallel to DEF and passing through B, $\overrightarrow{OB} \cdot \mathbf{n} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 42 < 56$ This implies the plane passing through B is between O and plane DEF. Hence B is beneath the canvas.	

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

(d)

Normal vector of the vertical wall is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $(d,0,0)$ lies on the vertical wall.

Equation of the vertical wall is

$$\begin{pmatrix}d\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d \Rightarrow \mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d$$

Direction vector of the line of intersection is

$$\begin{pmatrix}3\\3\\7\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\7\\-3\end{pmatrix}$$

Let $(x,y,0)$ be the common point lying on the two planes.

$$\begin{pmatrix}x\\y\\0\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56 \Rightarrow 3x+3y=56$$

$$\begin{pmatrix}x\\y\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=d \Rightarrow x=d$$

Solving the equations simultaneously,

$$3d+3y=56 \Rightarrow y=\frac{56}{3}-d$$

$$\therefore \mathbf{r}=\begin{pmatrix}d\\\frac{56}{3}-d\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\7\\-3\end{pmatrix}, \qquad \lambda\in\mathbb{R}$$

(e)

For P to shine the brightest at point B, P must be as near as possible to B. Thus, P is the foot of perpendicular from B to the roof.

Equation of the line passes through B and P:

$$\mathbf{r}=\begin{pmatrix}7\\7\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\3\\7\end{pmatrix}, \qquad \alpha\in\mathbb{R}$$

Sub $\overrightarrow{OP}$ into equation of roof,

$$\begin{pmatrix}7+3\alpha\\7+3\alpha\\7\alpha\end{pmatrix}\cdot\begin{pmatrix}3\\3\\7\end{pmatrix}=56$$

$$\alpha=\frac{14}{67}$$

$$\therefore \overrightarrow{OP}=\begin{pmatrix}7+3\alpha\\7+3\alpha\\7\alpha\end{pmatrix}=\frac{1}{67}\begin{pmatrix}469+3(67\alpha)\\469+3(67\alpha)\\7(67\alpha)\end{pmatrix}=\frac{1}{67}\begin{pmatrix}469+3(14)\\469+3(14)\\7(14)\end{pmatrix}$$

$$\overrightarrow{OP}=\frac{1}{67}\begin{pmatrix}511\\511\\98\end{pmatrix}$$

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.

Copyright © 2024/25
Sale or duplication is prohibited.

[Alternative method: find $\overrightarrow{BD}$, then find projection vector $\overrightarrow{BP} = \left(\overrightarrow{BD} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}$, then $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$]

Help protect against infringement. If you wish to provide a tip-off for any third-party copyright infringement of content **created by Indigo**, please contact us to receive a tip-off reward as our token of appreciation.